Lecture Note 4

Question 1

Minimize $x_1 + x_2 + x_3$

$x_1 + x_2 \geq 1$

$x_1 + x_3 \geq 1$

$x_2 + x_3 \geq 1$

Question 2 $\quad x_1 = x_2 = x_3 = 1/2$

Question 3 $\quad S = \{1,2,3\} \rightarrow$ not optimal as $S^* = \{1,2\}$

However, it satisfied the inequality as $\underbrace{|S|}_{3} \leq 2\underbrace{|S^*|}_{2}$

Question 4 $\quad$ We will have $S = \{1,2,3,4\} \rightarrow$ not optimal but $\underbrace{|S|}_{4} \leq 2\underbrace{|S^*|}_{2}$

Question 5 $\quad$ Input: Set V and $E \subseteq \{\{u,v\} : u,v \in V\}$, k

Question 6 $\quad$ Output: $S \subseteq V$

Question 8 $\quad$ Objective Function: Minimize $|S|$

Question 7 $\quad$ Constraint: $|\{\{u,v\} \in E : u \in S \text{ or } v \in S\}| \geq |E| - k$.

Question 9 $\quad$ % return YourOptimizationModel$(V, E, 0)$;

Question 10 $\quad$ As we can solve Vertex Cover by an algorithm, this problem is not easier than Vertex Cover. ~~As this proble~~ As Vertex Cover is NP-hard, this problem is also NP-hard.

Question 11 $\quad S = \{2,3\}$

Question 2.2

Minimize $x_1 + x_2 + x_3 + x_4 + x_5$

$x_1 + x_2 - r_1 \geq 0$

$x_1 + x_3 - r_2 \geq 0$

$x_2 + x_3 - r_3 \geq 0$

$x_2 + x_4 - r_4 \geq 0$

$x_3 + x_5 - r_5 \geq 0$

$r_1 + r_2 + r_3 + r_4 + r_5 \geq 4$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Question 2.3

$$A = \begin{bmatrix} \text{matrix in vertex cover} & -\text{identity matrix} \\ 0 \; 0 \; 0 \; \cdots \; 0 & 1 \; 1 \; \cdots \; 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ |E|-k \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Question 2.4

We have $x_1 = x_2 = x_3 = 1/3$. Then, we have $r_1 = r_2 = r_3 = \frac{2}{3}$.

$$r_1 + r_2 + r_3 \geq |E| - k = 3 - 1 = 2$$

Question 2.5

We will have $x_1' = x_2' = x_3' = 0 \rightarrow S = \emptyset$

Question 2.6

$$x_i' = \begin{cases} 0 & \text{if } x_i < \frac{1}{2} \cdot \frac{|E|-k}{|E|} \\ 1 & \text{otherwise.} \end{cases}$$

There will be at least $|E|-k$ members of $\{r_1, \ldots, r_{|E|}\}$ that $\geq \frac{|E|-k}{|E|} \cdot r_i$

Lecture Note 3

Q1 : Set V, Set $E \subseteq \{\{u,v\} : u, v \in V\}$

maximum # courses that we can recommend to a particular person k.

Q2 : For each $e \in E$, # courses recommend to e, x_e

Q3 : Maximize $\sum_e x_e$

Q4 : For all v, $\sum_{e : v \in e} x_e \le k$.

Q5 : A is an incident matrix of V, E, $b = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

We then have Maximize $c^T x$

s.t. $Ax \le b$.

Q6 : Solve linear program in Q5 to have $\begin{bmatrix} x_1 \\ \vdots \\ x_{|E|} \end{bmatrix}$

Have $x_i' = \begin{cases} k & \text{with prob. } x_i / 2|V|k \\ 0 & \text{otherwise} \end{cases}$

Q7 : The expected value of $\sum_{e : v \in e} x_e'$ is $\sum_{e : v \in e} k x_i / 2|V|k = \frac{k}{2|V|k} \sum_{e : v \in e} x_i \le \frac{k}{2|V|}$

By Markov's inequality, $Pr[\sum_{e : v \in e} x_e' > k = 2|V| \cdot EXP] = \frac{1}{2|V|}$.

Probability that each constraint is not satisfied is $\le \frac{1}{2|V|}$

Probability that 'some' constraints are not satisfied is $\le \sum_v \frac{1}{2|V|} = \frac{1}{2}$.

Q8 : Expected value of $\sum_e x_e'$ is $\sum_e k \cdot \frac{x_e}{2|V|k} = \frac{1}{2|V|} * \sum_e x_e$

$= \frac{1}{2|V|} \cdot OPT_{LP} \ge \frac{1}{2|V|} \cdot OPT$

Approximation ratio is $\frac{1}{2|V|}$.

Q9 : $S = \{1, 2\}$ $OPT = 2$

Q10 : Maximize $x_1 + x_2 + x_3 + x_4$

$x_1 + x_2 + x_3 \ge 1$

$x_1 + x_2 + x_4 \ge 1$

$x_1 + x_3 + x_4 \ge 1$

$x_2 + x_3 + x_4 \ge 1$

Q11 : $x_1 = x_2 = x_3 = x_4 = 1/3$

Q12 : 76

Q13 : $\{1, 2\}$

Lecture Note 6

Q1 Input: $\mathcal{A}$

Output: $\mathcal{B}$

Constraint: 1) $\mathcal{C}$

2) $\mathcal{D} \geq 0.5\,OPT$

Q2 Input: $\mathcal{A}$

Output: $\mathcal{B}$

Constraint: 1) $\mathcal{C}$

2) $1/\mathcal{D} \leq 2 \cdot OPT'$

Q3 return Problem 2 ($\mathcal{A}$);

We know that $OPT' = \frac{1}{OPT}$ as minimizing $\mathcal{D}$ is equivalent to maximizing $1/\mathcal{D}$.

Constraint 2) of Q1 $\boxed{\mathcal{D} \geq \frac{1}{2} OPT'} = \frac{1}{2\,OPT}$

$\boxed{2\,OPT \geq \frac{1}{\mathcal{D}}}$ constraint 2) of Q2

Hence, Problem in Q2 is equivalent to Q1.

Q4 If we have a 2-approximation algorithm for Problem 2, by Q3, we then also have a 0.5-approximation algorithm for 0.5-approximation algorithm for the problem in Q1. As we do not have the 0.5-approximation algorithm, we do not have the 2-approximation algorithm.

Q5.

OPT_{P1}	OPT_{P3}	Guarantee by 2-approximation algorithm	Guarantee for P_1
0	1	≤ 2	≥ 0
0.1	0.9	≤ 1.8	≥ 0
0.2	0.8	≤ 1.6	≥ 0
0.3	0.7	≤ 1.4	≥ 0
0.4	0.6	≤ 1.2	≥ 0
0.5	0.5	≤ 1	≥ 0
0.6	0.4	≤ 0.8	≥ 0.2
0.7	0.3	≤ 0.6	≥ 0.4
0.8	0.2	≤ 0.4	≥ 0.6
0.9	0.1	≤ 0.2	≥ 0.8
1	0	$\leq 0.$	≥ 1