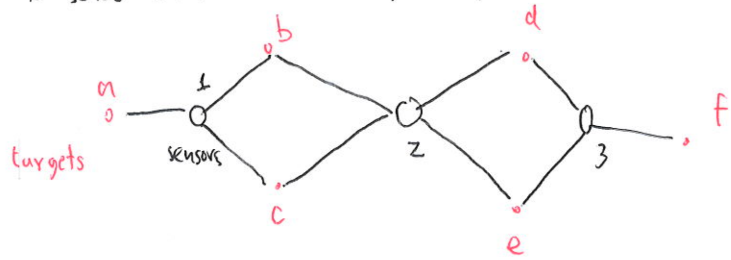
Sensor Network

Target Coverage Problem [Du and Wan 2013, Chapter2]

- We want to sense an information at specific places (such as widelifes) with smallest cost.



cost of sensor 1 $= y_1 = 6$

$y_2 = 9$

$y_3 = 15$

To cover all the targets with smallest cost

OPT $= \{1, 3\}$ cost of OPT $= y_1 + y_3 = 6 + 15$
$= 21$

Problem Definition

Input: # targets m [assume that the set of targets $= \{1, \dots, m\}$]

$S_1, \dots, S_n \subseteq \{1, \dots, m\}$: S_i is a set of targets that sensor i can monitor

$y_1, \dots, y_n$: y_i is a cost for sensor i

Output: A set of sensors that can cover all the target $C \subseteq \{1, \dots, n\}$

Constraint: $\bigcup_{j \in C} S_j = \{1, \dots, m\} \iff |\bigcup_{j \in C} S_j| = m$

Objective Function: Minimize $\sum_{j \in C} y_j$

Restate the problem [$f(C) = |\bigcup_{j \in C} S_j|$] ← f is monotone submodular.

Input: $f : 2^{\{1, \dots, n\}} \to \mathbb{Z}_{\geq 0}$, $y_1, \dots, y_n \in \mathbb{R}_{\geq 0}$
monotone submodular function

Output: $C \subseteq \{1, \dots, n\}$

Constraint: $f(C) = m = f(\{1, \dots, n\})$

Objective Function: Minimize $\sum_{j \in C} y_j$

Minimum Cost Submodular Cover Problem

Submodular Function Maximization with size constraint (1)

Maximize $f(C)$

s.t. $\underbrace{\text{cost}}_{|C|} \le k$

$\Longleftrightarrow$ primal-dual problem ?

Minimum Cost Submodular Cover Problem (2)

Minimize cost

s.t. $f(C) \geq m$.

However, solving (1) will not help (2) as the problem is not LP.

Algorithm

1: $P \leftarrow \phi$

2: Find sensor j that minimizes $\frac{y_j}{\Delta_j f(P)} = \frac{y_j}{f(P \cup \{j\}) - f(P)}$

→ cost we have to pay per one unit increment in function.

3: $P \leftarrow P \cup \{j\}$

4: If $f(P) = f(\{1, \dots, n\})$, terminates

Otherwise, Go to Step 2

Ex)

Step 1 $P \leftarrow \phi$

$\boxed{\text{Sensor } 1 \quad \frac{y_1}{\Delta_1 f(\phi)} = \frac{6}{f(\{1\}) - f(\phi)} = \frac{6}{3-0} = 2}$

Minimum

for the first 3 targets we pay 2 per each target

Sensor 2 $\frac{y_2}{\Delta_2 f(\phi)} = \frac{9}{4} = 2.25$

Sensor 3 $\frac{y_3}{\Delta_3 f(\phi)} = \frac{15}{3} = 5$

Choose sensor 1

$P \leftarrow P \cup \{1\} \rightarrow P = \{1\}$

Step 2 $P = \{1\}$

Sensor 1 $\dfrac{y_1}{\Delta_1 f(\{1\})} = \dfrac{6}{0} = \infty$

$\boxed{\text{Sensor 2} \quad \dfrac{y_2}{\Delta_2 f(\{1\})} = \dfrac{9}{2} = 4.5}$ Minimum.

for next 2 targets, we pay 4.5 each.

Sensor 3 $\dfrac{y_3}{\Delta_3 f(\{1\})} = \dfrac{15}{3} = 5$

Choose sensor 2!

Step 3 $P = \{1, 2\}$

Sensor 1 $\dfrac{y_1}{\Delta_1 f(\{1,2\})} = \infty$

Sensor 2 $\dfrac{y_2}{\Delta_2 f(\{1,2\})} = \infty$

$\boxed{\text{Sensor 3} \quad \dfrac{y_3}{\Delta_3 f(\{1,2\})} = \dfrac{15}{1} = 15}$ Minimum

for the last target we pay 15.

Choose sensor 3!

$P = \{1, 2, 3\}$ → cover all the targets

Output of the algorithm = SOL $= \{1,2,3\}$ cost of SOL $= 6+9+15 = 30$

OPT $= \{1,3\}$ cost of OPT $= 6+15 = 21$

Theorem $(\text{Cost of SOL}) \leq (\ln m + 1)(\text{Cost of OPT})$

More precisely $(\text{Cost of SOL}) \leq (\ln \Delta + 1)(\text{Cost of OPT})$

when $\Delta := \max_j \Delta_j f(\emptyset) = \max_j f(\{j\})$ = maximum # targets we can cover with 1 sensor $\leq m$

Recall Monotone function f: $f(S) \leq f(S')$ when $S \subseteq S'$

Submodular function f: $\Delta_e f(S) \geq \Delta_e f(S')$ when $S \subseteq S'$

Lemma (previously proved) $f(OPT) - f(S) \leq \sum_{j \in OPT} \Delta_j f(OPT)$.

Notation

j_t : element selected at the t^{th} iteration

P_t : set P after the t^{th} iteration

h : # iteration

Fact

1. $SOL = P_h$, $f(SOL) = f(P_h) = f(\{1, \dots, n\}) = f(OPT)$
2. j_t is the element that ~~maximizes~~ minimizes $\dfrac{y_j}{\Delta_j f(P_{t-1})}$
3. $P_t = \{j_1, \dots, j_t\}$

Notation $\mu_t = \dfrac{y_{j_t}}{\Delta_{j_t} f(P_{t-1})}$ (price for increasing f by one unit at iteration t)

Fact $\mu_t \leq \mu_{t+1}$ (we have to pay more on each iteration)

Proof $$\mu_t = \frac{y_{j_t}}{\Delta_{j_t} f(P_{t-1})} \leq \frac{y_{j_{t+1}}}{\boxed{\Delta_{j_{t+1}} f(P_{t-1})}} \leq \frac{y_{j_{t+1}}}{\Delta_{j_{t+1}} f(P_t)} = \mu_{t+1} \qquad \blacksquare$$

by the property of j_t

larger

Fact $y_{j_t} = \mu_t \Delta_{j_t} f(P_{t-1})$

Proof of the theorem

$$\text{Cost of } SOL = \text{Cost of } \{j_1, \dots, j_h\} = \sum_t y_{j_t}$$

$$= \sum_t \left[\mu_t \Delta_{j_t} f(P_{t-1}) \right]$$

$$= \sum_t \left[\mu_t \{ f(\underbrace{P_{t-1} \cup \{j_t\}}_{P_t}) - f(P_{t-1}) \} \right] = \sum_t \left[\mu_t \{ f(P_t) - f(P_{t-1}) \} \right]$$

$$= \sum_t \mu_t \{ [f(OPT) - f(P_{t-1})] - [f(OPT) - f(P_t)] \}$$

$$= \sum_{t=1}^{h} \mu_t [f(OPT) - f(P_{t-1})] - \sum_{t=1}^{h} \mu_t [f(OPT) - f(P_t)]$$

$$= \sum_{t=0}^{h-1} \mu_{t+1} [f(OPT) - f(P_t)] - \sum_{t=1}^{h} \mu_t [f(OPT) - f(P_t)]$$

$$= \mu_1 [f(OPT) - \underbrace{f(\cancel{P_0})}_{0}] + \sum_{t=1}^{h-1} \mu_{t+1} [f(OPT) - f(P_t)] - \sum_{t=1}^{h-1} \mu_t [f(OPT) - f(P_t)]$$

$$- \mu_h [\underbrace{\cancel{f(OPT)}}_{m} - \underbrace{f(P_h)}_{f(SOL) = m}]$$