Maximum Lifetime Coverage Problem

Input: $S_1, \dots, S_n$: a set of targets that can be monitor by each sensor.

Output: Suppose that $C_1, \dots, C_p$ are set of all target coverages.

Output $t_1, \dots, t_p$: time we use each target coverage.

Constraint: All sensors must be used for less than 1 hour

For all j, $\sum_{j \in C_i} t_i \le 1$

$\sum_i a_{j,i} t_i \le 1$ when $a_{j,i} = \begin{cases} 1 & j \in C_i \\ 0 & \text{otherwise.} \end{cases}$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,p} \\ a_{2,1} & \cdots & a_{2,p} \\ \vdots & & \\ a_{n,1} & \cdots & a_{n,p} \end{bmatrix} \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Objective Function: Maximize $\sum_i t_i = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^t \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix}$

Maximize $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^t \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix}$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,p} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,p} \end{bmatrix} \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\xLeftrightarrow{\text{Dual}}$

Minimize $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^t \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & & \vdots \\ a_{1,p} & \cdots & a_{n,p} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \ge \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Minimize $y_1 + \dots + y_n$

$a_{1,1} y_1 + \dots + a_{n,1} y_n \ge 1$

$\vdots$

$a_{1,p} y_1 + \dots + a_{n,p} y_n \ge 1$

Garg-Konemann Framework

1: $x \leftarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $y \leftarrow \begin{pmatrix} \delta \\ \delta \\ \vdots \\ \delta \end{pmatrix}$

2: Choose i^* that minimizes $A_i = a_{1,i} y_1 + a_{2,i} y_2 + \dots + a_{n,i} y_n$ → Minimum Cost Target Coverage.

3: If $A_{i^*} \ge 1$, terminates

4: For all j such that $a_{j,i^*} = 1$, $y_j \leftarrow y_j(1+\theta)$

5: $x_{i^*} \leftarrow x_{i^*} + \tau$

6: Go to Step 2

How to set τ? $\quad \tau := \dfrac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta}$

Theorem By the algorithm, the usages of all sensors are less than 1 hour.

Proof sensor i is used for τ hours

$\Leftrightarrow$ $i \in C_{j^*}$ where j^* is selected in the algorithm.

$\Leftrightarrow$ j^* minimizes $a_{1,j^*} y_1 + a_{2,j^*} y_2 + \dots + a_{n,j^*} y_n$

$\Leftrightarrow$ We increase the value of y_q for all q such that $a_{q,j^*} = 1$

$\Leftrightarrow$ We increase the value of y_q if $y_q \in C_{j^*}$

$\Leftrightarrow$ We increase the value of y_i

\# times that sensors i is used for τ hours = \# times y_i is multiplied by $(1+\theta)$

sensor i usage is τm. $\Leftrightarrow$ $y_i = \delta(1+\theta)^m$.

Suppose that sensor i is used for m times.

Before the last update $y_i = \delta(1+\theta)^{m-1}$

$$\underbrace{a_{i,j^*}}_{1} y_i \leq a_{1,j^*} y_1 + a_{2,j^*} y_2 + \dots + a_{n,j^*} y_n < 1$$

$$y_i < 1$$

$$\delta(1+\theta)^{m-1} < 1$$

$$(1+\theta)^{m-1} < 1/\delta$$

$$\ln((1+\theta)^{m-1}) < \ln 1/\delta$$

$$(m-1)[\ln(1+\theta)] < -\ln\delta$$

$$m-1 < \frac{-\ln\delta}{\ln(1+\theta)}$$

$$m < 1 - \frac{\ln\delta}{\ln(1+\theta)} = \frac{\ln(1+\theta) - \ln\delta}{\ln(1+\theta)}$$

$$\text{Sensor usage is } \tau m \leq \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta} \cdot \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta} = 1 \text{ hour.}$$ $\boxtimes$

Theorem Number of iterations $\leq$ ~~$\frac{\ln(1+\theta)}{\ln(1+}$~~ $\dfrac{n}{\tau} = \dfrac{n \cdot (\ln(1+\theta) - \ln\delta)}{\ln(1+\theta)}$

Proof The sum of sensor capacity at the beginning = n hours

Each iteration will use at least τ hours from the capacity sum.

\# iterations $\leq n/\tau$ $\boxtimes$

$$\min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} = y_1 + \dots + y_n \geq y_1^* + \dots + y_n^* = OPT$$

$$\therefore \quad OPT = \min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} \qquad \blacksquare$$

Lemma $\#\text{iterations} \geq \ln\left(\frac{1}{n\delta}\right) \frac{OPT}{\theta}$

Proof Let $y_i(t)$ be a value of y_i at Iteration t, and let $Y(t) = \sum_{i=1}^{n} y_i(t)$.

Recall that we select coverage C_{i^*} and multiply y_j with $(1+\theta)$ if $a_{j,i^*} = 1$.

$$Y(t+1) = \boxed{\text{terms that are not multiplied}} + (1+\theta) \cdot \boxed{\text{terms that are multiplied}}$$

$$= \boxed{\boxed{\text{terms that are not multiplied}} + \boxed{\text{terms that are multipied}}} \quad \text{all terms } Y(t)$$

$$+ \theta \cdot \boxed{\text{terms that are multiplied}}$$

$$= Y(t) + \theta \left[a_{1,i^*} y_1(t) + \dots + a_{n,i^*} y_n(t) \right]$$

$$= Y(t) + \theta \min_i \left[a_{1,i} y_1(t) + \dots + a_{n,i} y_n(t) \right] \qquad f((y_1(t), \dots, y_n(t))) \leq \frac{Y(t)}{OPT}$$

$$Y(t+1) = Y(t) + \theta \min_i \left[a_{1,i} y_1(t) + \dots + a_{n,i} y_n(t) \right] = Y(t) + \theta \cdot f((y_1(t), \dots, y_n(t)))$$

~~For~~ By previous lemma, $OPT = \min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} \leq \frac{\boxed{y_1(t) + \dots + y_n(t)}}{\boxed{f((y_1(t), \dots, y_n(t)))}}$ $Y(t)$

$$f((y_1(t), \dots, y_n(t))) \leq \frac{Y(t)}{OPT}$$

$$Y(t+1) \leq Y(t) + \theta \frac{Y(t)}{OPT} = Y(t)\left[1 + \frac{\theta}{OPT}\right] \leq Y(t)\, e^{\theta/OPT} \qquad 1 + x \leq e^x$$

$$Y(t) \leq e^{\theta/OPT \cdot t} \cdot Y(0) = e^{\theta/OPT \cdot t} \left[\overset{\delta}{y_1(0)} + \dots + y_n(0) \right]$$

$$= e^{\theta/OPT \cdot t} \cdot [\delta n]$$

Suppose that #iterations is T.

~~$\forall$~~ We know that
$$Y(T) = y_1(T) + \dots + y_n(T)$$
$$\geq a_{1,i} y_1(T) + \dots + a_{n,i} y_n(T) \quad \text{for all } i.$$
$$\geq 1$$

$$Y(T) \geq 1 \qquad Y(T) \leq e^{\theta/OPT \cdot T} [\delta n]$$

$$e^{\theta/OPT \cdot T} [\delta n] \geq 1$$

$$e^{\theta/OPT \cdot T} \geq 1/\delta n$$

$$\ln\left[e^{\theta/OPT \cdot T}\right] \geq \ln\left(\frac{1}{\delta n}\right)$$

$$\frac{\theta}{OPT} \cdot T \geq \ln\left(\frac{1}{\delta n}\right)$$

$$T \geq \frac{OPT}{\theta} \ln\left(\frac{1}{\delta n}\right) \qquad \blacksquare$$

How to assign δ? $\quad \delta := (1+\theta)\big((1+\theta)n\big)^{-1/\theta}$

Theorem $\quad \#\text{iterations} \le \underbrace{1/\theta^2}_{\text{constant}}\, n \cdot \ln n = O(n \ln n)$

Proof $\quad \#\text{iterations} \le n \cdot \dfrac{\ln(1+\theta) - \ln \delta}{\ln(1+\theta)}$

When θ is small,

$\ln(1+\theta) \approx \theta$

$$= n \cdot \frac{\ln(1+\theta) - \ln\left[(1+\theta)\big((1+\theta)n\big)^{-1/\theta}\right]}{\ln[1+\theta]}$$

$$= n \cdot \frac{\cancel{\ln(1+\theta)} - \cancel{\ln(1+\theta)} - \ln\big((1+\theta)n\big)^{-1/\theta}}{\cancel{\ln[1+\theta]}\ \theta}$$

$$= n \cdot \frac{1/\theta\, \ln(1+\theta)n}{\theta} = \frac{n}{\theta^2}\big(\underbrace{\ln(1+\theta)}_{\text{very small compared to } \ln n} + \ln n\big) = \frac{1}{\theta^2} n \ln n \quad \boxed{\checkmark}$$

Lemma Let OPT be the longest valid schedule's length (optimal value of primal problem)

$=$ smallest $y_1 + \dots + y_n$ (optimal value of dual problem)

For any $z \in (z_1, \dots, z_n) \in \mathbb{R}^n_{\geq 0}$, let

$$f(z) = \underbrace{\min_{i \in \{1,\dots,P\}} \left[a_{1,i} z_1 + a_{2,i} z_2 + \dots + a_{n,i} z_n\right]}$$

$\rightarrow$ smallest left side of dual constraints suppose to be ≥ 1

Then, $\displaystyle \min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} = OPT$

Result

$$\min\ y_1 + \dots + y_n \quad A^t \cdot y \geq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \longleftrightarrow \quad \min\left[\frac{z_1 + \dots + z_n}{f(z)}\right]$$

No constraint!

easy to conduct math analysis from here.

Proof Suppose that the optimal solution of the dual program is $\begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix}$ $\quad [OPT = y_1^* + \dots + y_n^*]$

$$\boxed{\begin{array}{c} a_{1,1} y_1^* + a_{2,1} y_2^* + \dots + a_{n,1} y_n^* \\ \vdots \\ a_{1,P} y_1^* + a_{2,P} y_2^* + \dots + a_{n,P} y_n^* \end{array}} \begin{array}{c} \geq 1 \\ \\ \geq 1 \end{array}$$

Minimum is 1

$f\big((y_1^*, \dots, y_n^*)\big) = 1$

$$\min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} \le \frac{y_1^* + \dots + y_n^*}{\underbrace{f\big((y_1^*, y_2^*, \dots, y_n^*)\big)}_{1}} = y_1^* + \dots + y_n^* = OPT$$

Next, $y_i = z_i / f(z)$ for all i.

$$\min_{i \in \{1,\dots,P\}} \left[a_{1,i} y_1 + a_{2,i} y_2 + \dots + a_{n,i} y_n\right] = \min_{i \in \{1,\dots,P\}} \left[a_{1,i} \frac{z_1}{f(z)} + a_{2,i} \frac{z_2}{f(z)} + \dots + a_{n,i} \frac{z_n}{f(z)}\right]$$

$$= \frac{1}{f(z)} \underbrace{\min_{i \in \{1,\dots,P\}} \left[a_{1,i} z_1 + a_{2,i} z_2 + \dots + a_{n,i} z_n\right]}_{f(z)}$$

$$= 1$$

$\therefore \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is a possible solution of the dual program